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VIRTUAL ENDOMORPHISMS OF NILPOTENT GROUPS

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ABSTRACT. A virtual endomorphism of a group G is a homomorphism $f: H \to G$ where H is a subgroup of G of finite index m. The triple (G, H, f) produces a state-closed (or, self-similar) representation φ of G on the 1-rooted m-ary tree. This paper is a study of properties of the image G^{φ} when G is nilpotent. In particular, it is shown that if G is finitely generated, torsion-free and nilpotent then G^{φ} has solvability degree bounded above by the number of prime divisors of m.

1. INTRODUCTION

A virtual endomorphism of a group G is a homomorphism $f : H \to G$ where H is a subgroup of G of finite index m. A recursive construction using f produces a so called *state-closed* (or, *self-similar*) representation of G on a 1-rooted regular m-ary tree. The kernel of this representation is the maximal subgroup K of H which is both normal in G and is f-invariant, in the sense that $K^f \leq K$; it is called the f-*core* (H).

The notion of virtual endomorphisms of groups is not recent. It already appeared in 1969, in M. Shub's [9], in connection with endomorphisms of compact differentiable manifolds. State-closed groups were introduced in [10], justified by the fact that the Grigorchuk 2-group, the Gupta-Sidki p-groups, the affine group $\mathbb{Z}^n GL(n, \mathbb{Z})$ [2], as well as an automata group of Aleshin- claimed to be free in [1]- satisfied such a condition. State-closed representations of groups on the binary tree were studied in some depth in [7] and dynamical aspects of these were developed by Nekrashevych into a far-reaching theory in [6].

The question of existence of finite-state, state-closed representations of certain groups, especially of free groups, stimulated a number of interesting constructions. Glasner and Mozes [4] used ideas from homogeneous tree lattices to obtain such a representation of a free group of rank 14 acting on on the 6-tree. This was followed by a construction by Muntyan and Savchuk (see [6] 1.10.3) for the free group of rank 2 on the 6-tree. More recently, Vorobets and Vorobets [12] have produced a proof that a group defined on the binary tree, related to the one proposed initially by Aleshin, is indeed free of rank 3.

The present paper extends the results on free abelian state-closed groups in [7] to finitely generated nilpotent state-closed groups. The main emphasis though is on the subclass of torsion-free groups- following P. Hall's notation, these are $\mathfrak{T}$-groups, or $\mathfrak{T}_c$-groups when the nilpotency class is c.

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We refer to the data $\{G,\, H \leq G,\, f : H \to G,\, [G : H] = m\}$ as a *triple* (G, H, f) of degree m. If the f-*core* (H) is the trivial subgroup then f and the triple (G, H, f) are called *simple* and when f is a simple epimorphism it is called *recurrent*. If the only f-invariant subgroup of G is the trivial subgroup then f and the triple (G, H, f) are called *strongly simple*. To give an example of a strongly simple triple, we let G be the free nilpotent group $F\,(c, d)$ of class c, freely generated by x_i $(1 \leq i \leq d)$, $H = \langle x_i^n \ (1 \leq i \leq d)\rangle$ where n is a fixed integer greater than 1 and let f be the extension of the map $x_i^n \to x_i$ $(1 \leq i \leq d)$. For another example, consider the group G of lower triangular matrices $\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix}$ with integer entries, H its subgroup of index 4, formed by the matrices $\begin{pmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 2c & b & 1 \end{pmatrix}$ and define

$$f : \begin{pmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 2c & b & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ -c & a & 1 \end{pmatrix}.$$

We review state-closed groups and representations in Section 2 and illustrate how to produce concretely state-closed nilpotent groups. Moreover, we prove that state-closed abelian groups are small, in the sense that their centralizers in the group of automorphisms of the tree coincide with their topological closure in this last group.

If G is an abelian group then naturally, $\ker(f) \leq f$-*core*(H) for any triple (G, H, f). The relationship between $\ker(f)$ and the f-*core*(H) for general nilpotent groups is established in Section 4.

Theorem 1. *Let G be a nilpotent group, H a subgroup of finite index m in G, $f \in Hom(H, G)$ and $L = f$-core(H). Then, $\ker(f) \leq \sqrt[H]{L}$, the isolator of L in H.*

A group G is said to be to be *compressible* provided every subgroup H of finite index in G contains a subgroup K isomorphic to G. It was shown by G. S. Smith in [11] that $\mathfrak{T}_c$-groups are compressible when $c \leq 2$. We extend this result to strongly simple triples in Subsection 5.1, as follows

Theorem 2. *Let G be a $\mathfrak{T}_c$-group with $c \leq 2$ and let H be a subgroup of finite index in G. Then there exists a subgroup K of finite index in H, which admits a strongly simple epimorphism $f : K \to G$.*

More on compressibility and co-Hopfianity questions concerning $\mathfrak{T}_c$-groups can be found in [3].

Given an integer $m > 1$, let $l\,(m)$ be the number of prime divisors of m (counting multiplicities) and $a\,(m)$ the largest exponent of the prime divisors of m. Denote by $c(G)$ the nilpotency class of G, by $s\,(G)$ the derived length of G and by $d(G)$ the minimum number of generators of G.

If G is a finitely generated nilpotent group and H a subgroup of G of index m then it is well-known that $\overline{G} = \frac{G}{core(H)}$ is finite and $c\left(\overline{G}\right) \leq a\,(m)$. We find in Subsection 5.2 such limitations for $s\,(G)$ and $c\,(G)$ with respect to $l\,(m)$.

Theorem 3. *Let G be a $\mathfrak{T}$-group and H a subgroup of finite index m in G. If $f : H \to G$ is simple then $s\,(G) \leq l\,(m)$. If f is strongly simple then $c\,(G) \leq l\,(m)$.*

The restriction on G is striking, given that $\mathfrak{T}_c$-groups have faithful finite-state representations on the binary tree (that is, $m = 2$) for any $c \geq 0$ (see, [7]). To show that in the first part of the theorem $s(G)$ cannot be replaced by $c(G)$, we construct in Subsection 5.4, an ascending sequence of simple triples (G_n, H_n, f_n) where the groups G_n are metabelian $\mathfrak{T}$-groups with $d(G_n) = 2$, $c(G_n) = n$, $[G_n : H_n] = 4$. Using another sequence of examples, we show that the limit in the second part of the theorem to be satisfactory.

It is important to observe that no such limitations exist for groups of prime power order. For let p be a fixed prime number, G be the s-iterated wreath product $W_s = (((C_p wr..)wr)C_p)\, wr C_p$, H its base subgroup and π_1 the projection of H on its 1st coordinate. Then $[G : H] = p$ and (G, H, π_1) is strongly simple, yet G has nilpotency class p^s and derived length $s + 1$.

In Section 6, we prove the following divisibility relation between indices of subgroups

Theorem 4. *Let G be finitely generated nilpotent group, H a subgroup of G of finite index $[G : H] = m$, $f : H \to G$ a monomorphism and $\left[G : H^f\right] = m'$. Furthermore, let U be a subgroup of H and $V = \left\langle U, U^f \right\rangle$. Suppose $[V : U] = l$, $\left[V : U^f\right] = l'$ are finite. Then there exist integers $m_1|m, m_1'|m'$ such that $lm_1' = l'm_1$.*

As an application, we obtain

Theorem 5. *Let G be a $\mathfrak{T}$-group, H a subgroup of G of finite index m which is a square-free integer and $f : H \to G$ a simple epimorphism. Then G is abelian.*

The combination of conditions $a(m) = 1$ and f being a simple epimorphism (recurrent) in the above theorem produced $s(G) = 1$; that is, $s(G) = a(m)$. This raises the question about possible improvements of the bound $l(m)$ in Theorem 3, under different types of conditions. Another question concerns the impact of the combination state-closed and finite-state would have on $\mathfrak{T}$-groups. For simple triples (G, H, f) where G is free abelian group of finite rank and $[G : H] = m = 2$, it was shown in [7] (see also, [6] Sec. 2.12) that the roots of the characteristic polynomial of f lie in the interior of the unit circle.

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2. State-closed groups and representations

Let Y be a non-empty set, $P(Y)$ the group of permutations of Y and let $\mathcal{T}(Y)$ be 1-rooted tree indexed by the free monoid Y^* generated by Y. Then the group of automorphisms $\mathcal{A} = Aut(\mathcal{T}(Y))$ of the tree is isomorphic to the semidirect product of $\mathcal{A}^Y$ by $P(Y)$ and under this identification, we have the decomposition $\mathcal{A} = \left(\mathcal{A}^Y\right) P(Y)$. Thus, an $\alpha \in \mathcal{A}$ is represented as $\alpha = (\alpha_y | y \in Y)\, \alpha^\sigma$ where $\alpha_y \in \mathcal{A}$ and $\sigma \in Hom(\mathcal{A}, P(Y))$.

The set of *states* of α is $Q(\alpha) = \{\alpha_u | u \in Y^*\}$. For $G \leq \mathcal{A}$, let $stab_G(i)$ denote the subgroup of G formed by elements which leave the ith level vertices fixed. The group $\mathcal{A}$ is the inverse limit of its quotients $\frac{\mathcal{A}}{stab_{\mathcal{A}}(i)}$ and as such becomes a topological group. Also, for $u \in Y^*$ of length i, let $\pi_u : stab_G(i) \to \mathcal{A}$ be the projection map on the uth coordinate. A subgroup G of $\mathcal{A}$ is *state-closed* provided $Q(\alpha)$ is a subset of G for all $\alpha \in G$ and is *finite-state* if $Q(\alpha)$ is finite for all $\alpha \in G$.

On fixing $y \in Y$, we obtain the triple $(G, stab_G(y), \pi_y)$. In the other direction, given a triple (G, H, f), we represent the group G on the right cosets of H, keeping track of the factor sets, as in Schreier's theorem. Then, we use f to repeat this process down a chain of subgroups. In the limit, this produces a state-closed representation of G on a rooted tree of degree $[G : H]$, as follows.

Theorem 6. *Let G be a group, H a subgroup of G, and Y a right transversal of H in G. Let σ be the permutation representation of G on Y; for $g \in G, y \in Y$, write $g^\sigma : Hy \to Hyg = Hy'$, $y' \in Y$ and $y' = (y)^{g^\sigma}$. Also, let $\mathcal{T}(Y)$ be 1-rooted tree indexed by the free monoid $Y^*$and let $f \in Hom(H, G)$. Then, the quadruple (G, H, Y, f) provides a representation φ of G into the automorphism group of the tree $\mathcal{T}(Y)$, defined by*

$$g^\varphi = \left\{ \left(yg. \left(y^{g^\sigma} \right)^{-1} \right)^{f\varphi} | y \in Y \right\} g^\sigma$$

Furthermore, $\ker(\varphi) = f$*-core*(H).

To dispel difficulties with the notation in the above formula, we offer a simple example. Let G be the additive group of integers $\mathbb{Z}$, $H = 2\mathbb{Z}$, and $Y = \{0, 1\}$; then, $\sigma : 0 \leftrightarrow 1$. Define $f : 2\mathbb{Z} \to \mathbb{Z}$ by $2n \to n$. Then, $1^\varphi = (0^\varphi, 1^\varphi)\,\sigma$ which is none other the binary adding machine.

The proof of the theorem is a direct extension of that of Theorem 3.1 in [7] and can be found in Section 2.5 of [6].

2.1. **Producing nilpotent state-closed groups.** We illustrate in this subsection how certain initial conditions about a simple triple (G, H, f) lead to an understanding of the properties of its state-closed representations, which in turn can be used to construct examples of such triples. Consider the following configuration:

$$[G : H] = 4,$$

$$\begin{aligned} z &\in Z(G), [\langle z \rangle : \langle z \rangle \cap H] = 2, \\ f &: H \to G \text{ such that } f : z^2 \to z. \end{aligned}$$

Let $Y = \{y_1, y_2, y_3, y_4\}$ be a right transversal of H in G. Then, we may choose $y_1 = e, y_2 = z$ and $y_4 = y_3 z$. We identify y_i with its subscript i. Thus, z^σ is the permutation $(1,2)(3,4)$ and $z^\varphi = (e, z^\varphi, e, z^\varphi) . (1,2)(3,4)$; we suppress φ from the notation and thus obtain

$$z = (e, z, e, z) . (1,2)(3,4).$$

Let $\mathcal{A} = Aut(\mathcal{T}(Y)), C = C_{\mathcal{A}}(z)$. We will show that

(i) C is state-closed;

(ii) the commutator equation $[\alpha, \beta] = z$ has the following particular solution in C:

$$\alpha = (\alpha, \alpha z, \alpha, \alpha)(1,2), \beta = \left(z, z, z^{-1}\beta, z^{-1}\beta\right)(1,3)(2,4);$$

Furthermore, The group $R = \langle \alpha, \beta \rangle$ is isomorphic to $F(2,2)$, is recurrent and is finite-state ;

(iii) there exists $\kappa \in N_C(R)$ defined by

$$\kappa = \left(\alpha^3\kappa^2, \alpha^3\kappa^2, \alpha\kappa^2, \alpha\kappa^2\right)$$

such that the group $S = \langle \alpha, \beta, \kappa \rangle$ is a $\mathfrak{T}_3$ group and has the presentation

$$\left\{ \alpha, \beta, \kappa | \, [\alpha, \beta, \alpha] = [\alpha, \beta, \beta] = [\alpha, \beta, \kappa] = [\alpha, \kappa] = e, [\beta, \kappa] = \alpha^{-2} \right\}.$$

Proof. It is straightforward to check that the elements of the centralizer $C = C_A(z)$ in $\mathcal{A}$ are of eight types:

$$\begin{aligned} x_1 &= (h_1, h_1, h_2, h_2), x_2 = (h_1, h_1 z, h_2, h_2)(1,2), \\ x_3 &= (h_1, h_1, h_2, h_2 z)(3,4), x_4 = (h_1, h_1 z, h_2, h_2 z)(1,2)(3,4), \\ x_5 &= (h_1, h_1, h_2, h_2)(1,3)(2,4), x_6 = (h_1, h_1 z, h_2, h_2 z)(1,4)(2,3), \\ x_7 &= (h_1, h_1, h_2, h_2 z)(1,3,2,4), x_8 = (h_1, h_1 z, h_2, h_2)(1,4,2,3) \end{aligned}$$

where in each case, $h_1, h_2 \in C$. Therefore, C is also state-closed.

If $x, x' \in C$ are such that $[x, x'] = z$ then there exist $x_2, x_5 \in \langle x, x' \rangle$ such that $[x_2, x_5] = z$. Let

$$x_2 = (h_1, h_1 z, h_2, h_2)(1,2), x_5 = (k_1, k_1, k_2, k_2)(1,3)(2,4).$$

Then $[x_2, x_5] = z$ if and only if

$$h_2 = k_1^{-1} h_1 k_1, [h_1, k_1 k_2] = z.$$

One solution is $h_1 = h_2 = x_2$, $k_1 = z$, $k_2 = z^{-1} x_5$. With these choices, rename x_2 as α and x_5 as β; thus,

$$\begin{aligned} \alpha &= (\alpha, \alpha z, \alpha, \alpha)(1,2), \\ \beta &= \left(z, z, z^{-1}\beta, z^{-1}\beta\right)(1,3)(2,4). \end{aligned}$$

It can be verified directly that α, β generate a group R isomorphic to $F(2,2)$, that R is finite-state, and recurrent.

We search in the normalizer of R in C for an element κ such that $\langle \alpha, \beta, \kappa \rangle$ is nilpotent of class 3. Clearly, we may assume $\kappa = (\kappa_1, \kappa_1, \kappa_2, \kappa_2)$. Since $\kappa^{-1}\kappa^{\alpha}, \kappa^{-1}\kappa^{\beta}$ stabilize the 1st level of the tree, there exist integers i, j, k, l, s, t such that

$$\kappa^{-1}\kappa^{\alpha} = \alpha^{2i}\beta^{2j}z^{r}, \kappa^{-1}\kappa^{\beta} = \alpha^{2l}\beta^{2k}z^{s}$$

and thus, modulo R', we have

$$\alpha^{\kappa} = \alpha^{1-2i}\beta^{-2j}, \beta^{\kappa} = \alpha^{-2l}\beta^{1-2k}.$$

Since the action of κ on R/R' is nilpotent, the matrix $\begin{pmatrix} 1-2i & -2j \\ -2l & 1-2k \end{pmatrix}$ has determinant 1 and trace 2. Thus,

$$k = -i, jl = -i^2.$$

Rather than describing all possible solutions we try

$$\begin{aligned} i &= j = k = 0, l = 1, \\ r &= s = 0. \end{aligned}$$

Then,

$$\begin{aligned} \kappa &= (\kappa_1, \kappa_1, \kappa_2, \kappa_2), \\ \kappa^{\alpha} &= \kappa, \kappa^{-1}\kappa^{\beta} = \alpha^2. \end{aligned}$$

Now, we calculate

$$\begin{aligned}
\kappa^{-1}\kappa^{\alpha} &= \left(\kappa_1^{-1}\kappa_1^{\alpha}, \kappa_1^{-1}\kappa_1^{\alpha}, \kappa_2^{-1}\kappa_2^{\alpha}, \kappa_2^{-1}\kappa_2^{\alpha}\right), \\
\kappa^{-1}\kappa^{\beta} &= \left(\kappa_1^{-1}\kappa_2^{\beta}, \kappa_1^{-1}\kappa_2^{\beta}, \kappa_2^{-1}\kappa_1, \kappa_2^{-1}\kappa_1\right), \\
\alpha^2 &= \left(\alpha^2 z, \alpha^2 z, \alpha^2, \alpha^2\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\kappa_1^{-1}\kappa_1^{\alpha} &= \kappa_2^{-1}\kappa_2^{\alpha} = e, \\
\kappa_1^{-1}\kappa_2^{\beta} &= \alpha^2 z, \kappa_2^{-1}\kappa_1 = \alpha^2,
\end{aligned}$$

$$\kappa_1 = \kappa_2\alpha^2, \kappa_2^{-1}\kappa_2^{\beta} = \alpha^4 z$$

and we find that $\kappa_2 = \alpha\kappa^2$ is a solution. Thus

$$\kappa = \left(\alpha^3\kappa^2, \alpha^3\kappa^2, \alpha\kappa^2, \alpha\kappa^2\right)$$

satisfies all our conditions.

The group $S = \langle \alpha, \beta, \kappa \rangle$ is a $\mathfrak{T}_3$ group with the presentation

$$\left\{\alpha, \beta, \kappa \mid [\alpha, \beta, \alpha] = [\alpha, \beta, \beta] = [\alpha, \beta, \kappa] = [\alpha, \kappa] = e, [\beta, \kappa] = \alpha^{-2}\right\}.$$

The groups R, S are the first two terms of an infinite sequence of nilpotent subgroups of $C_{\mathcal{A}}(z)$, which we will construct in Section 5.4.

2.2. **State-closed abelian groups.** Given $\alpha \in \mathcal{A}$ we indicate the diagonal automorphism $(\alpha, \alpha, ..., \alpha)$ by $\alpha^{(1)}$ and inductively, $\left(\alpha^{(i)}, \alpha^{(i)}, ..., \alpha^{(i)}\right)$ by $\alpha^{(i+1)}$.

The following theorem shows that recurrent abelian groups (no conditions on type) are in a sense small.

Theorem 7. *Let $Y = \{1, 2, ..., m\}$, $\mathcal{A} = Aut(\mathcal{T}(Y))$.*
(i) Let G be an abelian recurrent subgroup of $\mathcal{A}$ and $C_{\mathcal{A}}(G)$ be the centralizer of G in $\mathcal{A}$. Let $\widehat{G}$ the topological closure of G in $\mathcal{A}$. Then, $C_{\mathcal{A}}(G) = \widehat{G}$.
(ii) Let m be a prime number and G an infinite abelian state-closed subgroup of $\mathcal{A}$, which acts transitively on the $1st$ level of the tree. Then, $C_{\mathcal{A}}(G) = \widehat{G}$.

Proof. (i) Let the vertices of $\mathcal{T}(Y)$ be indexed by sequences from $Y = \{1, 2, ..., m\}$. Let G induce the permutation group P on the set Y. Then, P is an abelian transitive permutation group of degree m and is therefore regular; it follows that the stabilizer in G of any $y \in Y$ is the same as the stabilizer of the 1st level of the tree $H = stab_G(1)$. Since the representation of G is recurrent, the projection π_v of $stab_G(k)$ on any of its coordinates v produces the group G.

For every $\sigma \in P$, choose $a_{\mathbf{0}}(\sigma) = \left(a_{\mathbf{0}}(\sigma)_1, ..., a_{\mathbf{0}}(\sigma)_m\right)\sigma \in G$ which induces σ on Y. Let $h = (h_1, h_2, ..., h_m) \in H$. Then

$$h^{a_{\mathbf{0}}(\sigma)} = \left((h_1)^{a_{\mathbf{0}}(\sigma)_1}, (h_2)^{a_{\mathbf{0}}(\sigma)_2}, ..., (h_m)^{a_{\mathbf{0}}(\sigma)_m}\right)^{\sigma}$$

$$= (h_1, h_2, ..., h_m)^{\sigma}$$

since $h_i, a_{\mathbf{0}}(\sigma)_i \in G$ which is abelian. On varying $\sigma \in P$ we find that $h = (h_1, h_1, ..., h_1)$.

Now, for every $\sigma \in P$, there exists $a_{\mathbf{1}}(\sigma) = (a_{\mathbf{0}}(\sigma), ..., a_{\mathbf{0}}(\sigma)) \in H$, which induces $\sigma^{(1)}$ modulo $stab_{\mathcal{A}}(2)$. Thus, we produce a sequence $a_{\mathbf{i}}(\sigma) \in stab_G(i)$ of elements in G such that $a_{\mathbf{i}}(\sigma) = \sigma^{(i)}$ modulo $stab_{\mathcal{A}}(i+1)$.

Let $\gamma \in C = C_{\mathcal{A}}(G)$. Then,

$$\begin{aligned}\gamma &= (\gamma_1, ..., \gamma_m)\,\sigma, \\ \gamma' &= \gamma.a_{\mathbf{0}}(\sigma)^{-1} = (\gamma_1', ..., \gamma_m') \in stab_C(1)\end{aligned}$$

and $\gamma_1' = ... = \gamma_m'$; say γ_1' induces a permutation σ' on Y. Thus,

$$\gamma.a_{\mathbf{0}}(\sigma)^{-1}.a_{\mathbf{1}}(\sigma')^{-1} \in stab_C(2).$$

We produce in this manner a sequence

$$a_{\mathbf{0}}(\sigma), a_{\mathbf{1}}(\sigma'), a_{\mathbf{2}}(\sigma''), ...$$

of elements of G such that γ is equal to the infinite product

$$a_{\mathbf{0}}(\sigma)\, a_{\mathbf{1}}(\sigma')\, a_{\mathbf{2}}(\sigma'') ....$$

Hence, $C_{\mathcal{A}}(G) = \widehat{G}$.

(ii) Let $m = p$, a prime number. The permutation group P induced on $Y = \{1, ..., p\}$ is cyclic, say generated by σ. Since G is infinite, there exists an $h = (h_1, h_1, ..., h_1) \in H$ such that $h_1 \notin H$ and therefore we may assume h_1 induces σ on Y. We produce elements $a_{\mathbf{i}} \in G$ such that $a_{\mathbf{i}} = \sigma^{(i)}$ modulo $stab_{\mathcal{A}}(i+1)$ and the proof continues as previously. ∎

The group G in part (ii) need not be recurrent. For example, let $Y = \{1, 2\}$, G be the cyclic subgroup of $Aut(\mathcal{T}(Y))$ generated by $\alpha = (e, \alpha^3)\,\sigma$ where σ is the transposition $(1, 2)$.

Proposition 1. *Let G be a finitely generated abelian group, H a normal subgroup of G such that $\frac{G}{H}$ is cyclic of prime power order. Suppose (G, H, f) is a simple triple. Then, either G is finite or free abelian.*

Proof. Let $Tor(G)$ be the torsion subgroup of G. Then, we have the decompositions

$$G = Tor(G) \oplus K, H = Tor(H) \oplus M$$

where $Tor(H) \leq Tor(G)$ and we may assume $M \leq K$. Suppose G is a mixed group. Then the first possibility is $K = M$ and $\frac{Tor(G)}{Tor(H)}$ cyclic. Let n be the exponent of $Tor(G)$. Then, we have $(M^n)^f = (M^f)^n \leq G^n = M^n$; a contradiction. The other possibility is $Tor(G) = Tor(H)$ and $\frac{K}{M}$ finite cyclic ; but as $Tor(H)$ is f-invariant, $Tor(G) = Tor(H) = \{e\}$ and again we have a contradiction. ∎

3. Sub-triples and Quotient triples

A subgroup K of G is *semi-invariant* under the action of f provided $(K \cap H)^f \leq K$. If $K \leq H$ and $K^f \leq K$ then K is *f-invariant.* Given a triple (G, H, f) and $G_1 \leq G, H_1 \leq H \cap G_1$ such that $(H_1)^f \leq G_1$, we call $(G_1, H_1, f|_{H_1})$ a *sub-triple.* If N is a normal semi-invariant subgroup of G then $\overline{f} : \frac{HN}{N} \to \frac{G}{N}$ given by $\overline{f} : Nh \to Nh^f$ is well-defined and $\left(\frac{G}{N}, \frac{HN}{N}, \overline{f}\right)$ is a *quotient triple.*

Given a triple (G, H, f), we produce a sequence of subtriples $(G(i), H(i), f_i)$ defined as follows:

$$G(0) = G,\ H(0) = H,\ f_0 = f,$$

and for $i \geq 1$

$$G(i) = H(i-1)^f,\ H(i) = H(i-1) \cap G(i),\ f_i = f_{i-1}|_{H(i)}.$$

Clearly, if f is an epimorphism, then the sequence stops at $i = 0$.

Example 1. The following group G of automorphisms of the binary tree provides an example for which the sequence $G(i)$ is infinite,

$$G =< \alpha = \left(1, \alpha\beta^2\right)\sigma,\ \beta = (\alpha, \alpha) > .$$

It is straightforward to check that G is free abelian of rank 2, the subgroup $H =< \alpha^2, \beta >$ is of index 2 and the projection of H on the second coordinate is an extension of

$$f : \alpha^2 \to \alpha\beta^2, \beta \to \alpha.$$

We claim that for $i \geq 1$

$$G(i) =< \alpha^{2^{i-1}}, \alpha^{r_i}\beta^2 >$$

where

$$\begin{aligned} r_1 &= r_2 = 1, \\ r_i &= 1 + 4t_i \text{ such that } t_i r_{i-1} \equiv 1 \bmod \left(2^{i-2}\right) \text{ for } i \geq 3. \end{aligned}$$

The assertion is true for $i = 1, 2$:

$$\begin{aligned} G(0) &= G =< \alpha, \beta >, H(0) = H =< \alpha^2, \beta >; \\ G(1) &= H(0)^f =< \alpha\beta^2, \alpha >=< \alpha, \beta^2 >, \\ H(1) &= < \alpha^2, \beta^2 >; \\ G(2) &= H(1)^f =< \alpha\beta^2, \alpha^2, >=< \alpha^2, \alpha\beta^2 > . \end{aligned}$$

Now, suppose

$$G(i) =< \alpha^{2^{i-1}}, \alpha^{r_i}\beta^2 > .$$

Then

$$\begin{aligned} H(i) &= < \left(\alpha^{r_i}\beta^2\right)^2, \alpha^{2^{i-1}} >, \\ G(i+1) &= < \left(\alpha\beta^2\right)^{r_i}\alpha^4, \left(\alpha\beta^2\right)^{2^{i-2}} > . \end{aligned}$$

Viewing G as an additive group with basis α, β, the generators of $G(i+1)$ are the rows of the matrix $M = \begin{pmatrix} 4 + r_i & 2r_i \\ 2^{i-2} & 2^{i-1} \end{pmatrix}$.

Let m, k be integers such that $mr_i + k2^{i-2} = 1$ and let $S = \begin{pmatrix} 2^{i-2} & -r_i \\ m & k \end{pmatrix}$. Then, $\det(S) = 1$ and

$$SM = \begin{pmatrix} 2^i & 0 \\ m(4 + r_i) + k2^{i-2} & 2 \end{pmatrix}.$$

Since $m(4 + r_i) + k2^{i-2} = 4m + mr_i + k2^{i-2} = 4m + 1$, we reach

$$G(i+1) =< \alpha^{2^i}, \alpha^{1+4m}\beta^2 > .$$

Clearly, $[G : G(i)] = 2^i$ and therefore, $G(i) \neq G(j)$ for $i < j$.

Proposition 2. *Let G be group, H a subgroup G and suppose (G, H, f) is a simple triple such that $G = Z(G)H^f H$. Define,*

$$G(1) = H^f,\ H(1) = H \cap H^f,\ f_1 = f|_{H(1)}.$$

Then, $(G(1), H(1), f_1)$ is a simple triple.

Proof. Let Y be a right transversal of H in G such that Y is contained in $Z(G)H^f$. Let K be a subgroup of $H(1)$, normal in $G(1)$, with $K^f \leq K$. Then, $K \leq K^{f^{-1}}$ and $K^{f^{-1}}$ is normal in H. Then,

$$K = K^y \leq \left(K^{f^{-1}}\right)^y = \left(K^{f^{-1}}\right)^{hy}$$

for all $y \in Y$ and all $h \in H$; that is, $K \leq \left(K^{f^{-1}}\right)^g$ for all $g \in G$ Let $M = \cap_{g\in G} \left(K^{f^{-1}}\right)^g$. Then, $M \leq K^{f^{-1}} \leq H$, and M is normal in G such that

$$K \leq M \leq K^{f^{-1}},\ M^f \leq K \leq M;$$

therefore, $M = \{e\} = K$. ∎

The above result generalizes Lemma 3.2 in [7].

Lemma 1. *Let (G, H, f) be a triple. Suppose B is semi-invariant and $\sqrt[G]{B}$ is a group. Then $\sqrt[G]{B}$ is semi-invariant.*

Proof. Let $x \in \sqrt[H]{B}$, then $x \in H$ and $x^n \in B \cap H$. for some n. Therefore,

$$(x^n)^f = \left(x^f\right)^n \in (B \cap H)^f$$

and so, $x^f \in \sqrt[G]{(B \cap H)^f} \leq \sqrt[G]{B}$. ∎

3.1. **Facts about nilpotent groups.** We list below some facts about nilpotent groups which are either well-known (see [5],[8]) or have direct proofs. Let

$$\{Z_i(G) | 1 \leq i \leq c\},\ \{\gamma_i(G) | 1 \leq i \leq c\}$$

denote the upper and lower central series of G, respectively.

I. Let G be a nilpotent group of class c.

1. For all $1 \leq i, j \leq c$

$$\begin{aligned} \left[\gamma_i(G), \gamma_j(G)\right] &\leq \gamma_{i+j}(G). \\ \left[Z_i(G), \gamma_j(G)\right] &\leq Z_{i-j}(G). \end{aligned}$$

2. If $G = Z_i(G) H$ for some i, then for all $1 \leq j \leq c$,

$$\begin{aligned} \gamma_j(G) &\leq Z_{i-j+1}(G)\gamma_j(H), \\ \gamma_{i+1}(H) &= \gamma_{i+1}(G). \end{aligned}$$

3. The subset $Tor(G)$ of G of elements of finite order is a subgroup of G.
4. If $Z(G) \leq Tor(G)$ then $G = Tor(G)$.
5. Suppose N is a normal torsion-free subgroup of G. Let $x \in G$, $y \in N$ and n a positive integer. Then

$$[x^n, y] = e \Rightarrow [x, y] = e.$$

6. Suppose $Tor(G)$ has finite exponent s. If $G = Tor(G)K$ for some $K \leq G$. Then, $G^s = K^s$.

II. Let G be torsion-free nilpotent.

1. Let K be a subgroup of G. Then, the isolator of K in G

$$\sqrt[G]{K} = \{x \in G | x^n \in K \text{ for some positive integer } n\}$$

is a subgroup of G. If furthermore G is finitely generated then $\left[\sqrt[G]{K} : K\right]$ is finite.

2. Let H be a subgroup of finite index m in G. Then, $H \cap Z_i(G) = Z_i(H)$ for all i. Also, $[Z_i(G) : Z_i(H)] = q_i$ is finite for all i and q_i divides q_j for $i \leq j$.

III. Let G be finitely generated nilpotent group of class c.

1. Then G is Hopfian and has a finite Hirsch length denoted by $h(G)$. Also, a subgroup H has finite index in G if and only if $h(H) = h(G)$.

2. Let H be subgroup of finite index in G. Then $Z(G)$ and $Z(H)$ have the same Hirsch length. Also, $[\gamma_i(G) : \gamma_i(H)]$ is finite.

3.2. **Triples for nilpotent groups.**

Lemma 2. *Let G be a nilpotent group, (G, H, f) a triple, $\overline{G} = \frac{G}{Tor(G)}$ and $\overline{H} = \frac{HTor(G)}{Tor(G)}$. Then, $(Tor(G), Tor(H), f)$, $(\overline{G}, \overline{H}, \overline{f})$ are triples. Furthermore, if G is finitely generated and (G, H, f) is simple then $(\overline{G}, \overline{H}, \overline{f})$ is simple.*

Proof. The first assertion follows from $Tor(G) \cap H = Tor(H)$. Let $\frac{L}{Tor(G)} \leq \overline{H}$ be the $\overline{f}$-core$(\overline{H})$. Then,

$$L = Tor(G)(L \cap H) = Tor(L)(L \cap H)$$

and $(L \cap H)^f \leq L$. By Subsection 3.1, item I.6, there exists $s \geq 1$ such that $L^s = (L \cap H)^s$. Therefore, L^s is f-invariant. As f is simple, we have $L^s = \{e\}$, $L = Tor(G)$. ∎

Lemma 3. *Let G be a $\mathfrak{T}$ -group, H a subgroup of G of finite index in G and $f : H \to G$ a monomorphism. Then H^f has finite index in G. If U is an f-invariant normal subgroup of H then $U \cap Z(H)$ is an f-invariant normal subgroup of G. If (G, H, f) is simple then $(\langle H, H^f \rangle, H, f)$ is simple. .*

Proof. The groups H and G have equal Hirsch lengths, by Subsection 3.1, item I.4. Since $H^f \cong H$, it has the same Hirsch length as G and therefore H^f has finite index in G. Therefore, $Z(H), Z(H)^f \leq Z(G)$. Let $W = U \cap Z(H)$. Then,

$$W^f \leq U^f \cap Z(H)^f \leq U \cap Z(G) = W.$$

The last assertion follows directly. ∎

Proposition 3. *Let G be a $\mathfrak{T}_c$ -group, H a subgroup of finite index in G , $f \in Hom(H, G)$ be a simple monomorphism and $L \leq G$ be defined by $\frac{L}{Z(G)} = \overline{f}$-core$(\frac{HZ(G)}{Z(G)})$. Then*

(i) $\overline{f} : \frac{HZ(G)}{Z(G)} \to \frac{G}{Z(G)}$ induced by f is a monomorphism;

(ii) L, $\sqrt[G]{L}$ are abelian, semi-invariant and the corresponding quotient triples $\left(\frac{G}{L}, \frac{HL}{L}, \overline{f}\right), \left(\frac{G}{\sqrt[G]{L}}, \frac{H\sqrt[G]{L}}{\sqrt[G]{L}}, \overline{f}\right)$ are simple;

(iii) if $HL = H\sqrt[G]{L}$ then $L = \sqrt[G]{L}$;

(iv) if $G = H\sqrt[G]{L}$ then G is abelian.

Proof. Let $h \in H$ such that $Z(G)h \in \ker(\overline{f})$. Then, $h^f \in Z(G)$. As $Z(H)^f = Z(H^f) \leq Z(G)$ and $\ker(f) = \{e\}$, it follows that $h \in Z(H) \leq Z(G)$.

We have $Z(G) \leq L \leq Z(G)H$. Let $M = L \cap H$. Then, M is a normal subgroup of H and $L = Z(G) + M, L' = M'$. Also, $(Z(G)x)^{\overline{f}} = Z(G)x^f \in Z(G)M$ for all $x \in M$; that is, $M^f \leq Z(G)M$. Therefore $(M')^f \leq M'$. Since f is simple, M and L are abelian and therefore $\sqrt[G]{L}$ is abelian.

Write $\overline{G}=\frac{G}{Z(G)},\overline{L}=\frac{L}{Z(G)}$. As $\sqrt[G]{L}$ is abelian, easily, $\sqrt[\overline{G}]{\overline{L}}=\frac{\sqrt[G]{L}}{Z(G)}$. If $x\in\sqrt[H]{L}$ then there exists n such that $x^n\in L\cap H=M$; therefore $(x^n)^f=\left(x^f\right)^n\in L$ and $x^f\in\sqrt[G]{L}$. The assertion that $\left(\frac{G}{\sqrt[G]{L}},\frac{H\sqrt[G]{L}}{\sqrt[G]{L}},\overline{f}\right)$ is simple is now clear.

Let $\overline{G}=\frac{G}{L}$and $\overline{H}=\frac{HL}{L}$. Then $\frac{\sqrt[G]{L}}{L}=Tor\left(\overline{G}\right)$ and it follows that from Lemma 2 that $\left(\frac{G}{\sqrt[G]{L}},\frac{H\sqrt[G]{L}}{\sqrt[G]{L}},\overline{f}\right)$ is simple. Suppose $HL=H\sqrt[G]{L}$. Then, $\overline{H}=\overline{H}Tor\left(\overline{G}\right)$ and the equalities $Tor\left(\overline{H}\right)=Tor\left(\overline{G}\right)=\{L\}$ follow; that is, $L=\sqrt[G]{L}$.

Suppose $G=\sqrt[G]{L}H$. Then working modulo L, we have $\sqrt[G]{L}=T\left(G\right)$ and therefore there exists $s\geq 1$ such that $G^s=H^s$. In other words, going back to G, we have $G^sL=H^sL$. It follows from $L=Z(G)M$ that

$$[L,G^s]=[L,H^s]=[M,G^s]=[M,H^s]\,;$$

hence, $[M,H^s]$ is an f-invariant subgroup. Therefore,

$$[M,H^s]=[M,G^s]=\{e\}\,,M\leq Z(G)$$

and

$$L=Z(G)=\sqrt[G]{L},G=Z(G)H,G'=H'=\{e\}\,.$$

∎

We show in the next example that $\sqrt[G]{L}$ can be different from L.

Example 2. Let $G=F\left(2,2\right)$ freely generated by x_1,x_2. Let $n>1$ and let $H=<x_1^n,x_2^n>$. Then $Z\left(H\right)=\left\langle [x_2,x_1]^{n^2}\right\rangle$. The map $f:x_1^n\to x_1^n,x_2^n\to x_2$ extends to a monomorphism from H into G where $f:[x_2,x_1]^{n^2}\to[x_2,x_1]^n$. It is clear then that f is simple, $L=Z(G)<x_1^n>$ and $\sqrt[G]{L}=Z(G)<x_1>$.

4. Kernel versus Core

Proposition 4. *Let K,P be groups, P a transitive permutation group on the set $Y=\{1,...,m\}$ and P_1 be the stabilizer of 1 in P. Furthermore, let W be the wreath product Kwr_YP and let W act on Y as P. Let $B=K^Y$ and $W_1=BP_1$. Consider a nilpotent subgroup G of W which induces a transitive group on Y. Let $x=\left(x_1,x_2,...,x_m\right)\sigma\in G_1=G\cap W_1$. If x_1 has finite order then x also has finite order.*

Proof. Suppose that there exists an $x=\left(x_1,x_2,...,x_m\right)\sigma$ where $\sigma\in P_1$ such that x_1 has finite order, yet $o\left(x\right)$ is infinite. Then $x'=\left(x^{o(x_1)}\right)^{o(\sigma)}\in B_G=G\cap B$ and $x'=\left(e,x_2',...,x_m'\right)$ has infinite order. Let X be the set of non-trivial $x\in B_G$ such that each $x_i=e$ or $o\left(x_i\right)$ is infinite. Choose $x\in X$ such that first, x has a maximum number of trivial entries and second, $x\in\gamma_j\left(G\right)$ for a maximum j. Let $g=\left(g_1,g_2,...,g_m\right)\rho\in G$ where $\rho\in P$ and $o\left(\rho\right)=r$. Then, $[x,g^r]\in B_G$ has at least the same number of trivial entries as x and $[x,g^r]\in\gamma_{j+1}\left(G\right)$. Therefore, $[x,g^r]$ has finite order. Hence, in the torsion-free nilpotent group $\overline{G}=\frac{G}{Tor(G)}$, we have $[\overline{x},\overline{g^r}]=\overline{e}$ and so, $[\overline{x},\overline{g}]=\overline{e}$; that is, $[x,g]$ has finite order. We may assume $x_1=e$ and let i be such x_i has infinite order. Now let $g=\left(g_1,g_2,...,g_m\right)\rho\in G$ be such that $\left(i\right)\rho=1$. Then,

$$\begin{aligned}[x,g] &= x^{-1}x^g=\left(e,x_2^{-1},...,x_m^{-1}\right)\left(e,x_2^{g_2},...,x_m^{g_m}\right)^\rho\\ &= \left(x_i^{g_i},*,...,*\right)\end{aligned}$$

which has infinite order; a contradiction is reached. ∎

Theorem 8. *Let G be a nilpotent group, H a subgroup of finite index m in G, $f \in Hom\,(H,G)$ and $L = f$-core(H). Then,*
(i) $\ker(f) \leq \sqrt[H]{L}$;
(ii) $Tor\,(H)^f \leq Tor\left(H^f\right) \leq \left(\sqrt[H]{L}\right)^f$;
(iii) if $L = \{e\}$ *then* $Tor\,(H)^f = Tor\left(H^f\right)$;
(iv) if G *is finitely generated and* f *an epimorphism, then* $\sqrt[G]{L} = L$.

Proof. The triple (G,H,f) provides us with a state-closed representation $\varphi : \frac{G}{L} \to Aut\,(\mathcal{T}\,(Y))$, for $Y = \{1,...,m\}$ and where for $h \in H$, we have

$$h^{\varphi} = \left(h^{f\varphi}, *, ..., *\right)\sigma$$

and $(1)^{\sigma} = 1$.

(i) If $h \in \ker(f)$ then $h^{\varphi} = (e,*,...,*)\,\sigma$ and by the previous proposition, h^{φ} has finite order. As $L = \ker(\varphi)$, we have $h \in \sqrt[H]{L}$ and we are done.

(ii) The first inclusion is clear. Now suppose $x \in Tor\left(H^f\right)$; that is, $x = h^f$ and $x^n = e$ for some n. Then,

$$\begin{array}{rcl} e & = & (h^f)^n = (h^n)^f, h^n \in \ker(f)\,, \\ h & \in & \sqrt[H]{\ker(f)} \leq \sqrt[H]{L},\, x \in (\sqrt[H]{L})^f. \end{array}$$

If $L = \{e\}$ then $\sqrt[H]{L} = Tor\,(H)$ and the result follows from $Tor\,(H)^f \leq Tor\left(H^f\right) \leq Tor\,(H)^f$.

(iii) follows immediately from (ii).

(iv) Since G is finitely generated, $Tor(G)$ is a finite group. Suppose initially that L is trivial. Then, $\sqrt[G]{L} = Tor(G)$ and by item (ii), $Tor\,(H)^f \leq Tor\,(G) \leq Tor\,(H)^f$; thus,

$Tor\,(H)^f = Tor(G)$. As, $Tor\,(H) \leq Tor(G)$, we have $Tor\,(H) = Tor(G)$ and as f is simple, we conclude that $Tor(G)$ is trivial.

In the general case, we consider the triple $(\frac{G}{L}, \frac{H}{L}, \overline{f})$. Then, $\overline{f}$ is a simple epimorphism and therefore $Tor\left(\frac{G}{L}\right) = \{L\}$; that is, $\sqrt[G]{L} = L$. ∎

Corollary 1. *Let G be a torsion-free nilpotent group, H a subgroup of finite index m and $f{:}H \to G$ a homomorphism. Then*

$$f \textit{ simple} \Rightarrow \ker(f) = \{e\}\,.$$

Suppose $\ker(f) = \{e\}$. *Then*

$$f \textit{ simple} \Leftrightarrow f : Z(H) \to Z(G) \textit{ simple.}$$

Proof. The first assertion is a direct application of part (i) of the theorem. Suppose G is finitely generated; then $Z(H) \leq Z(G)$. It follows easily that f simple implies that $f|_{Z(H)} : Z(H) \to Z(G)$ simple. On the other hand, suppose $f : Z(H) \to Z(G)$ is simple and let K be a nontrivial subgroup of H, normal in G and f-invariant. Then, $K \cap Z(H) = \{e\}$ and

$$\begin{array}{rcl} (K \cap Z(H))^f & \leq & K \cap Z(H)^f = K \cap Z\left(H^f\right) \\ & \leq & K \cap Z\,(G) = K \cap H \cap Z\,(G) = K \cap Z(H). \end{array}$$

∎

Examples 3.

(1) *A simple triple* (G,H,f) *where* G *is finite and* $\ker(f)\neq\{e\}$.

Let p be a prime number, $Y=\{1,2,...,p\}$ and σ the permutation $(1,2,...,p)$. Let W_s be the group of automorphisms of the p-adic tree $\mathcal{T}(Y)$ generated by

$$\sigma_0=\sigma,\sigma_1=(e,...,e,\sigma_0),...,\sigma_s=(e,...,e,\sigma_{s-1}),$$

The W_s is the s-iterated wreath product $(((C_p wr..)wr)C_p)\,wrC_p(=W_{s-1}wrC_p)$. Let $H=stab_{W_s}(1)$ and $\pi_1:H\to W_s$. Then, $[W_s:H]=p$, $\pi_1(H)\cong W_{s-1}$ and $\ker(f)=\{e\}\times W_{s-1}\times...\times W_{s-1}$.

(2) *A simple triple* (G,H,f) *where* G *is of mixed type and* $\ker(f)\neq\{e\}$.

Let $G=(CwrD)\langle x\rangle$ where $C=\langle c\rangle, D=\langle d\rangle$ each of order p, and x of infinite order inducing conjugation by d on $CwrD$. Therefore, G is a nilpotent group with $Z(G)=\langle z,dx^{-1}\rangle$ where $z=cc^d..c^{d^{p-1}}$. Let $H=\langle C^D,x^p\rangle=\left\langle c,c^d,..,c^{d^{p-2}},z,x^p\right\rangle$ and $M=\langle d,x\rangle$ an abelian group of type $\mathbb{Z}_p\times\mathbb{Z}$. Then, H is abelian of type $(\mathbb{Z}_p)^p\times\mathbb{Z}$, $[G:H]=p^2$, $Z(H)=\langle z,x^p\rangle$. The extension of the map

$$c\to 1,c^d\to 1,...,c^{d^{p-2}}\to 1,z\to d,x^p\to x$$

produces an epimorphism $f:H\to M$. Then, $\ker(f)=\left\langle c,c^d,..,c^{d^{p-2}}\right\rangle$. Note that the only subgroup of $\ker(f)$ which is normal in $\langle c,d\rangle$ is the trivial subgroup. Let K be an f-invariant subgroup of H normal in G. Then,

$$K^f\leq K\cap M\leq H\cap M=\langle x^p\rangle.$$

Therefore, $K^f=\{e\},K\leq\ker(f)$ and so, $K=\{e\}$.

(3) *A triple* (G,H,f) *where* G *is* a $\mathfrak{T}$-*group and* f-*core*$(H)=Z(H)$.

Let $G=F(2,2)$ freely generated by a,b. Let H be the subgroup generated by a^3,b^2. Then, $[G:H]=36$. Define the endomorphism $f:H\to G$ extended from $f:a^3\to a^2,b^2\to b^3$. Then, $f:[a^3,b^2]\to[a^2,b^3]$. Thus, f-core$(H)=\left\langle [a,b]^6\right\rangle=Z(H)$.

5. Simple triples for $\mathfrak{T}$-groups

5.1. Nilpotent groups: 2-generated or of class 2.

Lemma 4. *Let* G *be a* $\mathfrak{T}_2$-*group. Then, there exists a subgroup* K *of* G *such that* $G=Z(G)K,K'=Z(K)$.

Proof. As $Z(G)$ is isolated in G, then $\frac{Z(G)}{G'}$ has a complement $\frac{K}{G'}$ in $\frac{G}{G'}$. Therefore,

$$\begin{aligned} G &= Z(G)K,G'=K', \\ G' &= Z(G)\cap K=Z(K). \end{aligned}$$

∎

Theorem 9. *Let* G *be a* $\mathfrak{T}_c$-*group with* $c\leq 2$ *and let* H *be a subgroup of finite index in* G. *Then there exists a subgroup* K *of finite index in* H, *which admits a strongly simple epimorphism* $f:K\to G$.

Proof. The case G abelian is obvious; so let G have nilpotency class 2. Choose $\{Z(G)a_1,...,Z(G)a_d\}$, a free generating set of $\frac{G}{Z(G)}$ and let $A=\langle a_1,a_2,...,a_d\rangle$. Then, by the previous lemma, $Z(A)=A'$ and $Z(G)=U_0\oplus U_1$ where $U_1=\sqrt[G]{Z(A)}$. Then, $G=U_0\oplus U_1A$.

There exists a generating set $\{x_1, x_2, ..., x_d\}$ of A such that modulo $Z(G)$, we have $H = \left\langle x_1^{k_1}, x_2^{k_2}, ..., x_d^{k_d} \right\rangle$ where $k_i \geq 1$. Thus, there exist $c_i \in Z(G)$ such that $b_i = c_i x_i^{k_i} \in H$ $(1 \leq i \leq d)$. Define $B = \langle b_1, b_2, ..., b_d \rangle$. Then,

$$\begin{aligned} H &= BZ(H), \\ Z(B) &= B' = \left\langle [x_i, x_j]^{k_i k_j} \,|i < j \right\rangle, \\ B' &\leq A' = \langle [x_i, x_j] \,|i < j \rangle. \end{aligned}$$

Let $V_1 = \sqrt[H]{Z(B)}$. Then, $V_1 \leq U_1$ and $[U_1 : V_1] = r_1$ is finite. Now, we prove that we may choose U_0 such that $H = V_0 \oplus V_1 B$ where $V_0 \leq U_0$. We argue in $\frac{Z(G)}{V_1}$. Since $\frac{U_1}{V_1} = Tor\left(\frac{Z(G)}{V_1}\right)$, there exists $W_0 \leq Z(G)$ such that $\frac{Z(G)}{V_1} = \frac{W_0 \oplus V_1}{V_1} \oplus \frac{U_1}{V_1}$ and $\frac{Z(H)}{V_1} \leq \frac{W_0 \oplus V_1}{V_1}$. It follows from $V_1 \leq Z(H) \leq W_0 \oplus V_1$ that $Z(H) = (Z(H) \cap W_0) \oplus V_1$. Let $[U_0 : V_0] = r_0$.

Now let $r = \operatorname{lcm}(r_0, r_1)$, $k = \operatorname{lcm}\{k_i | 1 \leq i \leq d\}$. Define the subgroups

$$\begin{aligned} B_0 &= \left\langle \left(c_i x_i^{k_i}\right)^{r \frac{k}{k_i}} |1 \leq i \leq d \right\rangle, \\ K &= Z(G)^{r^2 k^2} B_0. \end{aligned}$$

Then K is a subgroup of finite index in H and

$$K = U_0^{r^2 k^2} \oplus U_1^{r^2 k^2} B_0.$$

Now consider the map

$$\gamma : z \to z^{r^2 k^2}, x_i \to \left(c_i x_i^{k_i}\right)^{r \frac{k}{k_i}}.$$

We note that if the map γ extends to an endomorphism from G onto K then

$$\begin{aligned} \gamma &: U_0 \to U_0^{r^2 k^2}, U_1 \to U_1^{r^2 k^2}, \\ [x_i, x_j] &\to [x_i, x_j]^{r^2 k^2} \quad (1 \leq i \leq d). \end{aligned}$$

To prove that γ extends to an endomorphism, it is sufficient to observe that if for some $u_1 \in U_1$ and integer s we have u_1^s is a word $w([x_i, x_j])$ in the commutators $[x_i, x_j]$ then we have in the extension, $\gamma : u_1^s \to \left(u_1^{r^2 k^2}\right)^s = (u_1^s)^{r^2 k^2}$ on the one hand and $\gamma : w([x_i, x_j]) \to w\left([x_i, x_j]^{r^2 k^2}\right) = w([x_i, x_j])^{r^2 k^2}$ on the other and the two images coincide.

Example 4. Given a simple triple (G, H, f) where G is a $\mathfrak{T}$-group, a question may be posed as to whether assuming $f|_{Z(H)} : Z(H) \to Z(G)$ is an epimorphism implies that f itself is an epimorphism. The following is a counterexample: let $G = F(2,2)$, freely generated by a, b. Write $[a, b] = z$ and let $H_1 = \langle a^4, b^4, z^4 \rangle$, $H_2 = \langle a^2, b^2, z \rangle$. Then, $f : a^4 \to a^2, b^4 \to b^2, z^4 \to z$ extends to an isomorphism from H_1 onto H_2 and f is simple. Therefore, $Z(H_1)^f = Z(G)$, yet f is not an epimorphism. ∎

Lemma 5. *Let G is be a 2-generated $\mathfrak{T}_c$-group. Then $G' = Z_{c-1}(G)$.*

Proof. We have $G' \leq Z_{c-1}(G)$ and $\frac{G}{G'}, \frac{G}{Z_{c-1}(G)}$ are 2-generated non-cyclic abelian groups. Since $\frac{G}{Z_{c-1}(G)}$ is a is torsion-free quotient of $\frac{G}{G'}$, the result follows. ∎

Theorem 10. *Let G be 2-generated $\mathfrak{T}_c$-group. Suppose H is a proper normal subgroup of G of finite index m, which is isomorphic to G. Then, G is abelian.*

Proof. Let G be generated by a_1, a_2 and suppose $c \geq 2$.

First, we will argue the case $c = 2$. Then G is isomorphic to $F(2,2)$ and $G' = Z(G) =< [a_1, a_2] >$. We may choose the generators a_1, a_2 such that $HZ(G) =< a_1^{m_1}, a_2^{m_2} > Z(G)$. Thus,

$$H' = Z(H) =< [a_1, a_2]^{m_1 m_2} > .$$

We have from $[HZ(G), a_i]$ $(i = 1, 2)$

$$\begin{aligned} [a_1^{m_1}, a_2] &= [a_1, a_2]^{m_1},\ [a_1, a_2^{m_2}] = [a_1, a_2]^{m_2}, \\ [a_1, a_2]^{m_1}, [a_1, a_2]^{m_2} &\in H \cap Z(G) = Z(H). \end{aligned}$$

Therefore, $m_1 m_2 = \pm 1$. Hence, $HG' = G = H$.

Let $c \geq 2$ and $f : H \to G$ be an epimorphism. Then f induces an epimorphism $\overline{f} : \frac{HZ_{c-2}(G)}{Z_{c-2}(G)} \to \frac{G}{Z_{c-2}(G)}$, as $H \cap Z_{c-2}(G) = Z_{c-2}(H)$. The class 2 case leads to $G = HZ_{c-2}(G)$. Therefore

$$\frac{G}{Z_{c-2}(H)} = \frac{H}{Z_{c-2}(H)} \oplus \frac{Z_{c-2}(G)}{Z_{c-2}(H)}.$$

Moreover, since $\frac{G}{Z_{c-2}(H)}$ and $\frac{H}{Z_{c-2}(H)}$ are 2-generated, we reach $Z_{c-2}(G) = Z_{c-2}(H)$ and thus, $G = H$; again, a contradiction. ∎

5.2. **Derived length and nilpotency class.**

Theorem 11. *Let G be a $\mathfrak{T}$-group, H a subgroup of finite index $m > 1$ and $f : H \to G$ simple. Then $s(G) \leq l(m)$.*

Proof. We may suppose G non-abelian. By Corollary 1, f is a monomorphism. As H^f is a subgroup of finite index in G, we have $Z(H), Z(H)^f \leq Z(G)$. Clearly, $Z(G)$ is not contained in H; for otherwise, $Z(G) = Z(H)$ and f-invariant.

Consider the triple $\left(\frac{G}{Z(G)}, \frac{HZ(G)}{Z(G)}, \overline{f}\right)$. Then the index $\left[\frac{G}{Z(G)} : \frac{HZ(G)}{Z(G)}\right]$ is a proper divisor of m. Define $L \leq G$ by $\frac{L}{Z(G)} = \overline{f}$-core$\left(\frac{HZ(G)}{Z(G)}\right)$. By Proposition 3, both L, $\sqrt[G]{L}$ are abelian and the triples $\left(\frac{G}{L}, \frac{HL}{L}, \overline{f}\right), \left(\frac{G}{\sqrt[G]{L}}, \frac{H\sqrt[G]{L}}{\sqrt[G]{L}}, \overline{f}\right)$ are simple.

Now, we consider the chain of subgroups

$$H \leq HL \leq H\sqrt[G]{L} \leq G.$$

Since $HZ(G) \leq HL$ and $Z(G) \nleq H$, we have H is a proper subgroup of HL. By Proposition 3, if $HL = H\sqrt[G]{L}$ then $L = \sqrt[G]{L}$ and since G is non-abelian, $H\sqrt[G]{L} \neq G$.

We apply induction on $l(m)$. If $L = \sqrt[G]{L}$ then $\frac{G}{L}$ is torsion-free, $\left[\frac{G}{L} : \frac{HL}{L}\right] = m'$ and $l(m') < l(m)$; therefore $s\left(\frac{G}{L}\right) \leq l(m')$ and since L is abelian, $s(G) \leq l(m') + 1 \leq l(m)$. If $L \neq \sqrt[G]{L}$ then $HL \neq H\sqrt[G]{L}$ and therefore $\left[\frac{G}{\sqrt[G]{L}} : \frac{H\sqrt[G]{L}}{\sqrt[G]{L}}\right] = m''$, $l(m'') < l(m)$ and the argument proceeds as in the previous case. ∎

Corollary 2. *Let G be a finitely generated nilpotent group. Suppose $[G : H] = p$ a prime number and (G, H, f) a simple triple. Then, G is a finite p-group (no restriction on nilpotency class or derived length) or is free abelian.*

Proof. Proceed by induction on the order of $Tor(G)$. If G is torsion-free then by the previous theorem, G is abelian.

If $Tor(G) \neq \{e\}$ then $Tor(G)$ is not contained in H. Since H is a maximal subgroup of G, by Proposition 2, $\left(H^f, H \cap H^f, f\right)$ is a simple triple of degree p. Since $\left|Tor(H^f)\right| < |Tor(G)|$ we conclude that H^f is finite or torsion-free. In the first case, it follows that G is finite. In the second case, $\ker(f) = Tor(H)$, normal in G and f-invariant; therefore, $\ker(f) = \{e\}$ and G is torsion-free, contrary to the assumption.

To justify that neither $s(G)$ nor $c(G)$ can be bounded in case G is finite, we recall W_s, the iterated wreath product of cyclic groups of order p in Examples 3 (1). ∎

Theorem 12. *Let G be a $\mathfrak{T}$-group, H a subgroup of finite index m. Suppose $f : H \to G$ is strongly simple. Then $\overline{f} : \frac{HZ(G)}{Z(G)} \to \frac{G}{Z(G)}$ is also strongly simple. Furthermore, $c(G) \leq l(m)$.*

Proof. Let G be non-ablelian. Suppose there exists a nontrivial subgroup $\frac{L}{Z(G)}$ of $\frac{Z(G)H}{Z(G)}$ which is $\overline{f}$- invariant; then there exists $K \leq H$ such that $L = Z(G)K$; thus $L' = K'$ and is f-invariant. Since f is strongly simple, L is free abelian of finite rank and we may assume $L = Z(G) \oplus K$, as $Z(G)$ is isolated. The decomposition of L provides us with uniquely defined homomorphisms

$$\zeta \in Hom\left(K, Z(G)\right), \gamma \in Hom\left(K, K\right)$$

defined by $k^f = k^\zeta + k^\gamma$ for all $k \in K$. We note that γ is a monomorphism whose characteristic polynomial μ is non-constant..

Since $[L : Z(H) \oplus K]$ is finite, f extends to an automorphism $\widehat{f}$ of $\mathbb{Q} \otimes L$, say having characteristic polynomial ρ; clearly, μ is a factor of ρ. However, since f is simple, ρ does not have monic integral polynomial factors of positive degree; a contradiction.

Induction on $c(G)$ leads directly to a proof of the last assertion. ∎

The following natural example shows that the above limit is satisfactory.

Example 5. Let V_n be the additive free $\mathbb{Z}$-module of rank n generated by v_i $(1 \leq i \leq n)$ and let $x_n \in GL(V_n)$ be defined by

$$\begin{aligned} x_n \quad &: \quad v_i \to v_i + v_{i+1} \quad (1 \leq i \leq n-1), \\ v_n \quad &\to \quad v_n. \end{aligned}$$

Also, let G_n be the semidirect product $G_n = V_n \langle x_n \rangle$. Then G_n is a $\mathfrak{T}_n$-group. Let p be a prime number, $W_n = pV_n$, $x' = v_n x$ and $H_n = W_n \langle x'_n \rangle$. Then, $[G_n : H_n] = p^n$ and

$$f_n : pv_i \to v_i \ \ (1 \leq i \leq n), \ x'_n \to x_n$$

extends to a strongly simple epimorphism $f : H \to G$.

5.3. **Triples with degree a product of two primes.** Let (G, H, f) be a simple triple of degree $m = pq$, a product of two primes. In contrast to the prime degree, here we have a greater variety of groups. However, we do not know of examples of simple (G, H, f) where G is a non-abelian $\mathfrak{T}$-group, $m = pq$ and p, q distinct primes. We show below that G can be a mixed non-abelian group.

Examples 6. (1) *A mixed nilpotent group G of class 2, with $[G : H] = p^2$.*

Let $G =< a,u,v \mid u^p = v^p = [u,v] = e, u^a = uv, v^a = v >$ of type $(\mathbb{Z}_p \times \mathbb{Z}_p)\,\mathbb{Z}$. Let $H =< u, a^p >$ and $K =< v, a >$. Then $[G : H] = p^2$ and H, K are abelian of type $\mathbb{Z}_p \times \mathbb{Z}$.

Define $f : H \to K$ by $f : u \to v, a^p \to a,$. Then, f is a simple homomorphism; for

$$u^{a^p} = uv^p = u, f : u^{a^p} (= u) \to v^a (= v) .$$

(2) *A mixed nilpotent group G of class 2, with $[G : H] = pq$ where p is a prime, $q = 1 + tp$ and (G, H, f) simple.*

Let $D = \left\langle a, b | a^{p^2} = b^p = e, a^b = a^{1+p} \right\rangle$, a group of order p^3. Let $G = D. \langle x \rangle$ where x is of infinite order and acts on D as conjugation by b. Then,

$$Z\,(G) = \left\langle a^p, b^{-1}x \right\rangle .$$

We observe that $\left(b^{-1}x\right)^p = x^p$. Let $q = 1 + tp$ and let

$$H = \langle a^p, b, x^q \rangle \,, K = \langle b, x \rangle .$$

Then, H is abelian, has index pq in G, has type $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}$ and K is abelian of type $\mathbb{Z}_p \times \mathbb{Z}$. Moreover,

$$H \cap Z\,(G) = \left\langle a^p, b^{-1}x^q \right\rangle$$

since $b^{-1}x. \left(x^p\right)^t = b^{-1}x^q$. Now the map

$$f : a^p \to b, b \to e, x^q \to x$$

extends to an epimorphism $f : H \to K$. To prove that f is simple we observe that

$$\begin{aligned} f \quad &: \quad H \cap Z\,(G) \to \langle b, x \rangle \,, \\ H \cap \langle b, x \rangle \quad &= \quad \langle b, x^q \rangle \,, \\ f \quad &: \quad \langle b, x^q \rangle \to \langle x \rangle \,. \end{aligned}$$

Theorem 13. *Let G be a $\mathfrak{T}_c$-group, (G, H, f) a simple of degree pq where p, q are (not necessarily distinct) prime numbers and let $\frac{L}{Z(G)} = \overline{f}$-core $\left(\frac{HZ(G)}{Z(G)}\right)$. Then,*
(i) L and $\frac{G}{L}$ are free abelian groups;
(ii) $G = HH^f$;
(iii) $Z_{c-1}(G) \leq L$;
(iv) $Z(G) = \sqrt[G]{\gamma_c\,(G)}$.

Proof. Suppose G is non-abelian. Since $Z(H) \neq Z(G)$, we may suppose $[G : HZ(G)] = p, [HZ(G) : H] = q$. Then, $[Z(G) : Z(H)] = q$ and $HZ(G) = HL$.

The subgroups $HZ(G)$, $(HZ(G))' = H'$ are normal in G. Since $(\frac{G}{L}, \frac{HL}{L}, \overline{f})$ is a simple triple of degree a prime, by Corollary 2, $\frac{G}{L}$ is either free abelian or finite. We know that L and $\sqrt[G]{L}$ are abelian. Therefore, $\frac{G}{L}$ is free abelian.

As $Z(H)^f \not\leq Z(H)$ and $[Z(G) : Z(H)] = q$, we obtain $Z(G) = Z(H)Z(H)^f$. It follows from $(HZ(G))\, H^f = HH^f$ that HH^f is a subgroup of G and $HH^f = HZ(G)$ or G. Suppose the first alternative holds. Then, $\left(HH^f\right)' = H'$, a normal subgroup of G and $(H^f)' = (H')^f \leq H'$. Therefore, H is abelian and hence, central; this leads to $[G : Z(G)] = p$ which is absurd.

Consider the first index j such that $Z_j(G) \not\leq L$ and define $K = Z_j(G)L$. We assert that $\gamma_i\,(K) \leq Z_{j-i+1}(G)$ for $i \geq 2$:

$$\gamma_2\,(K) = Z_j(G)'\,[L, Z_j(G)] \leq Z_{j-1}(G) \leq L;$$

if $\gamma_i(K) \leq Z_{j-i+1}(G)$ for some i then

$$\begin{aligned}\gamma_{i+1}(K) &\leq [Z_{j-i+1}(G), Z_j(G)L] \\ &\leq [Z_{j-i+1}(G), Z_j(G)]\,[Z_{j-i+1}(G), L] \\ &\leq Z_{j-i+2}(G).\end{aligned}$$

Since $(L \cap H)^f \leq L$, and $(Z_j(G), Z_j(H), f)$ is a triple, we have the corresponding triple $(\frac{K}{L}, \frac{LZ_j(H)}{L}, \overline{f})$. Furthermore, as $\frac{G}{L}$ is free abelian and $(\frac{G}{L}, \frac{LH}{L}, \overline{f})$ is simple of degree p, it follows that $\frac{K}{L}$ is of finite index in $\frac{G}{L}$ and therefore, K and G have the same nilpotency class; that is, G has class $j = c$. Hence, $G = Z_c(G) = K$ and $Z_{c-1}(G) \leq L$.

Since $(Z(G), Z(H), f)$ is a simple triple of prime degree and $(\gamma_c(G), \gamma_c(H), f)$ is a sub-triple, it follows that $[Z(G) : \gamma_c(G)]$ is finite; hence $Z(G) = \sqrt[G]{\gamma_c(G)}$. ∎

5.4. **A sequence of simple triples of degree 4.** The groups R, S produced in Subsection 3.2 will be shown to be part of an ascending sequence of simple triples (G_n, H_n, f_n) where $[G_n : H_n] = 2^2$, $d(G_n) = 2$, $s(G_n) = 2$ and $c(G_n) = n$. This will prove that the nilpotency class of groups in Theorem 12 cannot have a fixed upper limit.

Let V_n be the free $\mathbb{Z}$ module $\mathbb{Z}^n$ and $\{\varepsilon_i | 1 \leq i \leq n\}$ its canonical basis. Define inductively $x_n \in GL(V_n)$:

$$\begin{aligned}x_2 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \\ x_n &= \begin{pmatrix} 1 & \xi_n \\ 0 & x_{n-1} \end{pmatrix}, \xi_n = \left(2^{n-2}, 0, ..., 0\right)\end{aligned}$$

for $n \geq 3$. Then x_n acts nilpotently and uniserially on V_n. We note that x_n^2 leaves invariant the submodule $W_n = \langle \varepsilon_1, .., \varepsilon_{n-2}, \varepsilon_{n-1} + \varepsilon_n, 2\varepsilon_n \rangle$, where clearly, $[V_n : W_n] = 2$. Define the semidirect product $G_n = V_n \langle x_n \rangle$ and its subgroup $H_n = W_n \langle x_n^2 \rangle$. Then, G_n is nilpotent of class n and $Z(G_n) = \langle \varepsilon_n \rangle$. Also, $[G_n : H_n] = 2^2$ and $Z(H_n) = \langle 2\varepsilon_n \rangle$. Furthermore, $G_2 \cong R, G_3 \cong S$.

We construct inductively simple endomorphisms $f_n : W_n \to V_n$:

$$\begin{aligned}f_2 &= \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}, \\ f_n &= \begin{pmatrix} 2^{n-2} & \theta_n \\ 0 & f_{n-1} \end{pmatrix}, \theta_n = (\theta_{n,1}, ..., \theta_{n,n-1})\end{aligned}$$

for $n \geq 3$.

We extend f_n, first by stipulating that $f_n : x_n^2 \to x_n$. Then, f_n extends to a homomorphism $H_n \to G_n$ if and only if $x_n^2 f_n = f_n x_n$. It is easy to see that f_2 satisfies this equation. For $n \geq 3$, $x_n^2 f_n = f_n x_n$ is equivalent to

$$\theta_n (1 - x_{n-1}) = \xi_n \left(2^{n-2} - (1 + x_{n-1}) f_{n-1}\right).$$

This last equation provides a unique solution θ_n where

$$\begin{aligned}\theta_{n,1} &= 2^2\theta_{n-1,1} + 2^{2n-6},\\ \theta_{n,i} &= 2^{i+1}\left(\theta_{n-1,i} + 2^{n-4}\theta_{n-2,i-1}\right) \quad \left(2 \le i \le \left[\frac{n}{2}\right]\right)\\ &= 0 \quad \left(\lfloor \frac{n+1}{2} \rfloor \le i \le n-1\right).\end{aligned}$$

The referee provided the following explicit solution

$$\theta_{n,i} = \frac{n}{i}\binom{n-i-1}{i-1} 2^{\left(n-\frac{i}{2}-3\right)(i+1)+1}.$$

The first few vectors are

$$\begin{aligned}\theta_3 &= (3,0), \theta_4 = \left(2^4, 2^2, 0\right),\\ \theta_5 &= \left(2^4 5, 2^4 5, 0, 0\right), \theta_6 = \left(2^7 3, 2^7 3^2, 2^8, 0, 0\right).\end{aligned}$$

Finally, the resulting f_n is simple. For otherwise, if K is a non-trivial subgroup of H_n, which is normal in G_n and is invariant under f_n then $\{e\} \neq Z(H_n) \cap K$ would also be f_n-invariant.

6. Index Theorem and an Application

Let G be a $\mathfrak{T}_c$-group, H a subgroup of finite index m. and $f : H \to G$ simple. Then, $\left[G : H^f\right] = m'$, $[\gamma_c(G) : \gamma_c(H)] = l$, $\left[\gamma_c(G) : \gamma_c\left(H^f\right)\right] = l'$ are finite. The following general result establishes a simple arithmetic relation between m, m', l, l'.

Theorem 14. *Let G be finitely generated nilpotent group, H a subgroup of G of finite index $[G : H] = m$ and $f : H \to G$ a monomorphism, also let $\left[G : H^f\right] = m'$. Furthermore, let U be a subgroup of H and let $V = \left\langle U, U^f\right\rangle$. Suppose $[V : U] = l, \left[V : U^f\right] = l'$ are finite. Then there exist integers $m_1 | m, m_1' | m'$ such that $lm_1' = l'm_1$.*

Before giving the proof, we note that the theorem is clearly true for finite groups, since $|H| = \left|H^f\right|, |U| = \left|U^f\right|$. However, the following example shows that the theorem is not valid for the class of 2-generated metabelian groups.

Example 7. Let p be a prime number and P be the subgroup of the additive rationals generated by $\left\{p^i | i \in \mathbb{Z}\right\}$. Then P admits the automorphism $f : p^i \to p^{i-1}$. Define G to be the extension of P by $\langle f\rangle$; then G is generated by $\{1, f\}$. Now let $H = G$ and $U = \langle p\rangle \le P$. Then, $U^f = \langle 1\rangle = V$. Moreover, the indices are $[G : H] = 1 = \left[G : H^f\right], [V : U] = p, \left[V : U^f\right] = 1$.

Proof. I. Suppose G is a free additive abelian group.

Then,

$$\sqrt[G]{V} = \sqrt[G]{U} = \sqrt[G]{U^f}$$

and $\left[\sqrt[G]{U} : U\right] = t, \left[\sqrt[G]{U^f} : U^f\right] = t'$ are finite. Indeed, $t = t'$, as f induces an isomorphism between the quotient groups $\frac{\sqrt[H]{U}}{U}, \frac{\sqrt[H^f]{U^f}}{U^f}$.

Define the indices

$$\left[H + \sqrt[G]{V} : H\right] = m_1, \left[H^f + \sqrt[G]{V} : H^f\right] = m_1'$$

where $m_1 | m$ and $m_1' | m'$. Since

$$H \cap \sqrt[G]{V} = \sqrt[H]{U}, H^f \cap \sqrt[G]{V} = \sqrt[H^f]{U^f}$$

we conclude

$$\left[\sqrt[G]{V}:\ \sqrt[H]{U}\right]=m_1,\left[\sqrt[G]{V}:\ \sqrt[H^f]{U^f}\right]=m_1'.$$

Now, we calculate the index

$$\left[\sqrt[G]{V}:U\cap U^f\right]=\left[\sqrt[G]{V}:\ \sqrt[H]{U}\right]\left[\sqrt[H]{U}:U\right]\left[U:U\cap U^f\right]=$$
$$\left[\sqrt[G]{V}:\ \sqrt[H^f]{U^f}\right]\left[\sqrt[H^f]{U^f}:U^f\right]\left[U^f:U\cap U^f\right].$$

Thus, $m_1tl'=m_1'tl$ and we reach $m_1l'=m_1'l$.

We have $Tor\left(H\right)^f=Tor\left(H^f\right)\le Tor(G)$. The map $\overline{f}:\frac{Tor(G)H}{Tor(G)}\to\frac{G}{Tor(G)}$ where $Tor(G)h\to Tor(G)h^f$ is a well-defined monomorphism. Since $\frac{G}{Tor(G)}$ is torsion-free nilpotent, it follows that $\left[\frac{G}{Tor(G)}:\frac{Tor(G)H^f}{Tor(G)}\right]$ is finite and therefore $\left[G:H^f\right]=m'$ is finite.

II. Suppose $Tor\left(G\right)=\{e\}$ We proceed by induction on the nilpotency class of G. The case where G is free-abelian was done in part I.

We introduce the following notation

$$\begin{aligned}Z_H &= H\cap Z\left(G\right),Z_{H^f}=H^f\cap Z\left(G\right)\\ [G:HZ] &= m_2,[HZ:H]=m_1,\left[G:H^fZ\right]=m_2',\left[H^fZ:H^f\right]=m_1'.\end{aligned}$$

Then,

$$\begin{aligned}Z\left(H\right)^f &= \left(H\cap Z\left(G\right)\right)^f=\left(Z_H\right)^f,\\ [Z:Z_H] &= m_1,[Z:Z_{H^f}]=m_1',\,m=m_2m_1,m'=m_2'm_1'.\end{aligned}$$

Similarly, with respect to V, denote

$$Z_V=V\cap Z,Z_U=U\cap Z,Z_{U^f}=U^f\cap Z,$$

and

$$[V:UZ_V]=l_2,[UZ_V:U]=l_1,\ \left[V:U^fZ_V\right]=l_2',\left[U^fZ_V:U^f\right]=l_1'.$$

Then

$$[Z_V,Z_U]=l_1,[Z_V,Z_{U^f}]=l_1',\,l=l_2l_1,l'=l_2'l_1'.$$

We claim that $\left(Z_U\right)^f=Z_{U^f}$. This follows from

$$\left(Z_U\right)^f\le Z\left(H^f\right)\le Z\left(G\right),\left(Z_U\right)^f\le U^f\cap Z\left(G\right)=Z_{U^f}$$

and from

$$\begin{aligned}Z_{U^f} &\le Z\left(H^f\right)=Z\left(H\right)^f=\left(Z\left(G\right)\cap H\right)^f,\\ Z_{U^f} &\le \left(Z\left(G\right)\cap H\right)^f\cap U^f=\left(Z\left(G\right)\cap U\right)^f=\left(Z_U\right)^f.\end{aligned}$$

From the configuration

$$Z\ge Z_H,Z_{H^f},\,Z_V\ge Z_U,Z_{U^f}$$

we obtain that there exist $m_{11}|m_1,m_{11}'|m_1'$ such that $l_1m_{11}'=l_1'm_{11}$.

Next, we apply induction to the nilpotency class of G.

We have $Z(H)\le Z\left(G\right)$ and $Z(H)^f=Z\left(H^f\right)\le Z\left(G\right)$. Moreover, $\overline{f}:\frac{Z(G)H}{Z(G)}\to\frac{G}{Z(G)}$ defined by $Z\left(G\right)h\to Z\left(G\right)h^f$ is a monomorphism and $(\frac{Z(G)U}{Z(G)})^{\overline{f}}=\frac{Z(G)U^f}{Z(G)}$. By applying induction to the class of $\frac{G}{Z(G)}$, we obtain that there exist $m_{21}|m_2,m_{21}'|m_2'$ such that $l_2m_{21}'=l_2'm_{21}$. Hence, putting together the two equations $l_1m_{11}'=l_1'm_{11},l_2m_{21}'=l_2'm_{21}$, we obtain

$$m_1' = m_{11}'m_{21}'|m', m_1 = m_{11}m_{21}|m$$

$$\begin{aligned} l_1 m_{11}' l_2 m_{21}' &= l_1' m_{11} l_2' m_{21}, \\ l m_1' &= l' m_1. \end{aligned}$$

III. Now we argue the general case where $T = Tor(G)$ is not necessarily trivial. Similar to the work done in part II, we define $T_H = T \cap H$ and likewise we define $T_{H^f}, T_V, T_U, T_{U^f}$, all finite groups. We note also that $(T_U)^f = T_{U^f}$ and therefore T_U, T_{U^f} have equal orders. Then it follows that

$$[V : UT_V] = [T_V : T_U] = [T_V : T_{U^f}] = \left[V : U^f T_V\right].$$

Finally, the argument continues as in part (II) with T substituting Z. ∎

A special case of the above result is

Corollary 3. *Maintain the hypotheses of the theorem. (i) If $U^f \leq U$ then $l = \left[U : U^f\right]$ is finite and $l|m$. (ii) If $U \leq U^f$ then $l' = \left[U^f : U\right]$ is finite and $l'|m'$.*

Proof. Suppose $U^f \leq U$. Then as U^f is isomorphic to U, we conclude $\left[U : U^f\right]$ is finite. The remaining assertions are direct. ∎

We use the above divisibility criterion to prove

Theorem 15. *Let G be a $\mathfrak{T}_c$-group, H a subgroup of finite index m in G and $f : H \to G$ an epimorphism. Let $[G : HZ_{c-1}(G)] = k$, $[Z(G) : Z(H)] = q$. Then, $[\gamma_c(G) : \gamma_c(H)]$ is a k-number which divides q. If f is simple and m a square-free integer then G is abelian.*

Proof. We may assume $c \geq 2$. By Subsection 3.1, item I.2, if $G = HZ_{c-1}(G)$ then $\gamma_c(G) = \gamma_c(H)$. So, suppose $G \neq HZ_{c-1}(G)$ and consider the free abelian group $\overline{G} = \frac{G}{Z_{c-1}(G)}$. There exist $a_1, a_2, ..., a_s \in G$ such that

$$Z_{c-1}(G)\,a_1, Z_{c-1}(G)\,a_2, ..., Z_{c-1}(G)\,a_s$$

freely generate $\overline{G}$ and integers $k_1|k_2|...|k_r$, $k = k_1k_2...k_r$ such that

$$Z_{c-1}(G)\,a_1^{k_1}, Z_{c-1}(G)\,a_2^{k_2}, ...Z_{c-1}(G)\,a_r^{k_r}, Z_{c-1}(G)\,a_{r+1}.., Z_{c-1}(G)\,a_s$$

freely generate $\overline{H} = \frac{HZ_{c-1}(G)}{Z_{c-1}(G)}$. Thus, there exist $c_1, c_2, ..., c_s \in Z_{c-1}(G)$ such that

$$H = \left\langle c_1 a_1^{k_1}, c_2 a_2^{k_2}, ..., c_r a_r^{k_r}, c_{r+1} a_{r+1}.., c_s a_s \right\rangle Z_{c-1}(H).$$

Now, $\gamma_c(G) \leq Z(G)$ and is generated by simple commutators $[a_{i_1}, a_{i_2}, ..., a_{i_c}]$ of weight c where the indices i_j are from $\{1, 2, ..., s\}$. Whereas, $\gamma_c(H)$ is generated by $[a_{i_1}, a_{i_2}, ..., a_{i_c}]^{\lambda(i_1,...,i_c)}$ where $\lambda(i_1, ..., i_c) = k_1^{u_1}...k_t^{u_t} \neq 1$ and u_z is the number of $i_j = z \in \{1, ..., r\}$. Therefore, $\left|\frac{\gamma_c(G)}{\gamma_c(H)}\right|$ is a k-number.

As f induces epimorphisms $\gamma_c(H) \to \gamma_c(G)$, $Z(H) \to Z(G)$, we apply Corollary 3 to obtain $\left|\frac{\gamma_c(G)}{\gamma_c(H)}\right|$ divides $\left|\frac{Z(G)}{Z(H)}\right| = q$.

Now suppose f is simple, m square-free and G non-abelian. Since f is simple, we have $Z_{c-1}(G) \nleq H$. Let

$$[G, HZ_{c-1}(G)] = m_1, [HZ_{c-1}(G), H] = m_2;$$

then, $\gcd(m_1, m_2) = 1$. Since $[Z(G) : Z(H)]$ divides m_2, we conclude that $\gamma_c(G) = \gamma_c(H)$ and therefore, $\gamma_c(G) = \{e\}$; a contradiction. ∎

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